

Quantum uncertainties in coupled harmonic oscillator

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Abstract

In this paper we analyze the quantum uncertainties and the photon statistics in the interaction between the two modes of radiation by treating them as coupled harmonic oscillator with the motivation of controlling quantum properties of one light beam by another. Under the rotating wave approximation (RWA) we show that if initially one of the modes is coherent and the other one squeezed, then the squeezing and non-Poissonianness of the photon statistics can transfer from one mode to the other. We give a parametric study of these properties depending upon interaction time and the degree of initial squeezing in one of the modes.

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Squeezed states of quantum systems have been an active area of interest for more than a decade [1]. Nonclassical states of radiation fields showing squeezing properties have been experimentally produced in four wave mixing procedures or by passing a coherent beam through optically nonlinear medium [2]. The attempt to generate squeezed radiation is still on as a challenging technological problem [3]. These quantum features have many potential applications in interferometry [4] and in noise free transmission of information [5].

An important goal for the future is the development of all-optical control and communication systems in which one light beam is controlled by another lightbeam. These two mode type of interactions can be treated by two coupled harmonic oscillators interacting through coupling [6]. As we are interested in quantum properties of light, it becomes interesting to explore the nature of uncertainties involved in the interaction of two coupled oscillators. The classical solution of the coupled harmonic oscillator shows the transfer of energy between the two oscillators. From this fact, the obvious question arises that what happens to the other non-classical properties (squeezing, non-Poissonian statistics). In this paper we calculate the transfer of the non-classical properties from one mode to the other through the interaction (coupling) between two radiation fields. For experimental purpose, we consider a dual channel coupler (fig.1) [7]. It consists of a pair of optical waveguides which run in sufficiently close proximity, for a certain distance, so that coherent coupling takes place between them. The coupling characteristics is sensitively dependent upon the refractive index difference between the guides, and in non-centrosymmetric nonlinear material, such as LiNbO₃, this can be controlled through the Pockel's effect. The coupling characteristics can be controlled by an external electrical signal.

The Hamiltonian of the coupled harmonic oscillator is given by (in units of \hbar)

$$H = H_0 + V \tag{1a}$$

where $H_0 = \omega_1 a_1^\dagger a_1 + \omega_2 a_2^\dagger a_2$ describes the free Hamiltonian and $V = g(a_1^\dagger a_2 + a_2^\dagger a_1)$ is

the interaction or coupling term under rotating wave approximation (RWA) with strength of the order of g . $\omega_i (= 2\pi\nu_i)$ are the measure of the frequencies ν_i and $a_i(a_i^\dagger)$ are the annihilation (creation) operators for the two modes. Choosing the central energy of the oscillators $\hbar\omega_0(\omega_0 = \frac{\omega_1+\omega_2}{2})$ to be zero and defining $\delta = \frac{\omega_1-\omega_2}{2}$, the above equation reduces to

$$H = \delta(a_1^\dagger a_1 - a_2^\dagger a_2) + g(a_1^\dagger a_2 + a_2^\dagger a_1) \quad (1b)$$

Solving the Heisenberg equation of motion for the creation and annihilation operators for the two modes and the above hamiltonian, we find their time evolution to be

$$\begin{bmatrix} a_1(t) \\ a_2(t) \end{bmatrix} = \begin{bmatrix} \mathcal{A}_1 & \mathcal{A}_2 \\ \mathcal{A}_2 & \mathcal{A}_1^* \end{bmatrix} \begin{bmatrix} a_1(0) \\ a_2(0) \end{bmatrix} \quad (2a)$$

and

$$\begin{bmatrix} a_1^\dagger(t) \\ a_2^\dagger(t) \end{bmatrix} = \begin{bmatrix} \mathcal{A}_1^* & -\mathcal{A}_2 \\ -\mathcal{A}_2 & \mathcal{A}_1 \end{bmatrix} \begin{bmatrix} a_1^\dagger(0) \\ a_2^\dagger(0) \end{bmatrix} \quad (2b)$$

where, $\mathcal{A}_1 = \cos(\Omega t) - i\frac{\delta}{\Omega} \sin(\Omega t)$ and, $\mathcal{A}_2 = -i\frac{g}{\Omega} \sin(\Omega t)$ with $\Omega = \sqrt{\delta^2 + g^2}$.

To calculate the different matrix elements of the observables showing nonclassical properties for the two modes we define our product oscillator state

$$|\Psi\rangle = |\psi_1\rangle \otimes |\psi_2\rangle = |\alpha_1, \xi\rangle \otimes |\alpha_2\rangle \quad (3)$$

where the first oscillator is in squeezed coherent state $|\alpha_1, \xi\rangle$ and the second one is in coherent state $|\alpha_2\rangle$. α_i are complex coherence parameters of the two modes and $\xi = re^{i\phi}$ is the squeezing parameter of the first mode. Using the results of time evolution of the creation and annihilation operators and setting ξ to be real and equal to be r , we calculate the uncertainties in the quadratures in the two modes (first index is for mode while the second index is for quadrature) for the state $|\Psi\rangle$

$$\Delta X_{11}^2 = \frac{1}{2} + |\mathcal{A}_1|^2 \sinh r [\sinh r + \cosh r \cos 2\theta_{\mathcal{A}_1}] \quad (4a)$$

$$\Delta X_{12}^2 = \frac{1}{2} + |\mathcal{A}_1|^2 \sinh r [\sinh r - \cosh r \cos 2\theta_{\mathcal{A}_1}] \quad (4b)$$

$$\Delta X_{21}^2 = \frac{1}{2} + |\mathcal{A}_2|^2 \sinh r [\sinh r + \cosh r \cos 2\theta_{\mathcal{A}_2}] \quad (4c)$$

$$\Delta X_{22}^2 = \frac{1}{2} + |\mathcal{A}_2|^2 \sinh r [\sinh r - \cosh r \cos 2\theta_{\mathcal{A}_2}] \quad (4d)$$

where, $\theta_{\mathcal{A}_i}$ are the arguments of \mathcal{A}_i respectively. The terms in the square brackets can in general be negative according to the choice of parameters. We have plotted their time evolution in fig.2 for $|\alpha_1| = |\alpha_2| = 5.0$ and, $g = \frac{\delta}{10}$ with different r to show transfer of squeezing. Figs. 2(a-b) shows oscillation in the uncertainties of the quadratures with the frequency Ωt . Without interaction this would have been $2t$ [8]. So, by controlling the interaction strength and/or the frequency difference between the oscillator the first oscillator can be controlled. The plots in figs. 2(c-d) show that the squeezing is achieved in one quadrature of the second oscillator while the other quadrature is antisqueezed throughout the time which is not seen in single mode squeezing. However, they oscillate in time with the same frequency Ωt and come back to the coherent state periodically. In the case of single mode squeezing the uncertainty ellipse rotates in the phase space in time, but here the uncertainty circle deforms to ellipse and return back to circle in time. The amount of squeezing increases with the degree of squeezing of the first mode (r) as expected. The squeezing can be generated in the other quadrature of the second mode by simply setting $\xi = -r$ ($\phi = \pi$).

We have also calculated the mean and variances of the number operators for both the modes. The mean numbers are given by

$$\langle n_1 \rangle = |\mathcal{A}_1 \nu|^2 + |\mathcal{A}_1 \alpha_1 + \mathcal{A}_2 \alpha_2|^2 \quad (5a)$$

$$\langle n_2 \rangle = |\mathcal{A}_2 \nu|^2 + |\mathcal{A}_2 \alpha_1 + \mathcal{A}_1 \alpha_2|^2 \quad (5b)$$

It is trivial to put the expressions of \mathcal{A}_i and show that the mean numbers oscillate confirming the transfer of number of photons i.e. energy from one mode to the other. However, the

total number operator \hat{N} ($=\sum a_i^\dagger a_i$) remains invariant over time. Due to the interest about the statistics of the photon numbers in both the modes we have calculated the variances in the number as

$$\Delta n_1^2 = [|\mathcal{A}_1\nu|^2 + |\mathcal{A}_1\alpha_1 + \mathcal{A}_2\alpha_2|^2] + |\mathcal{A}_1|^2 \sinh r [\sinh r - 2|\mathcal{A}_1\alpha_1 + \mathcal{A}_2\alpha_2|^2 \{\sinh r - \cosh r \cos 2\theta_{\mathcal{A}_1\alpha_1 + \mathcal{A}_2\alpha_2}\}] \quad (6a)$$

$$\Delta n_2^2 = [|\mathcal{A}_2\nu|^2 + |\mathcal{A}_2\alpha_1 + \mathcal{A}_1\alpha_2|^2] + |\mathcal{A}_2|^2 \sinh r [\sinh r - 2|\mathcal{A}_2\alpha_1 + \mathcal{A}_1\alpha_2|^2 \{\sinh r - \cosh r \cos 2\theta_{\mathcal{A}_2\alpha_1 + \mathcal{A}_1\alpha_2}\}] \quad (6b)$$

where, $\theta_{\mathcal{A}_1\alpha_1 + \mathcal{A}_2\alpha_2}$ and $\theta_{\mathcal{A}_2\alpha_1 + \mathcal{A}_1\alpha_2}$ are the phases of $(\mathcal{A}_1\alpha_1 + \mathcal{A}_2\alpha_2)$ and $(\mathcal{A}_2\alpha_1 + \mathcal{A}_1\alpha_2)$ respectively. Note that the number uncertainty of the first mode is no longer time-independent as in the case of a single squeezed radiation mode. Though the number uncertainty can show squeezing, but we are not interested in that at present as it has importance in context of phases of the modes.

The important features in the calculation of the number variables is not in the result of the mean or the uncertainties, but in the difference between them (we call it bunching parameter $\langle \mathcal{B}_i \rangle = \Delta n_i^2 - \langle n_i \rangle$), which is a measure of the statistics of the number of photons in the different modes. If $\langle \mathcal{B}_i \rangle$ is positive or negative, then the statistics of that mode (i) follows super- or sub-Poissonian statistics and that mode is called to be bunched or anti-bunched. In figs. 3(a-b) we plot the time evolution of the bunching parameter ($\langle \mathcal{B}_i \rangle$). The bunching parameter of the first mode (fig.3a) is no longer a constant difference between constant variance and constant mean, but oscillates in time. The oscillator was initially chosen to be bunched ($\langle \mathcal{B}_1 \rangle > 0$) and the oscillation is small enough to maintain it to be bunched. But the bunching parameter of the second mode shows both antibunching ($\langle \mathcal{B}_2 \rangle < 0$) and bunching ($\langle \mathcal{B}_2 \rangle > 0$) as $\langle \mathcal{B}_2 \rangle$ oscillates over time. The amount of antibunching is seen to be more than bunching for the second mode. Also the time spent as antibunched is greater than the time spent as bunched.

In conclusion, we have calculated the noises in the quadratures and the bunching parameter for both the modes as a function of time to show that the sharing of the non-classical properties of a squeezed harmonic oscillator to a coherent (classical) one is possible when they are coupled. The procedure is supported by an experimental scheme. The experimental situation can also be achieved where two radiations of different frequencies interact through a medium consisting of three-level atoms. The energy difference of the allowed transitions of the atom should be resonant to that of the radiations or vice versa. The interaction effect generated by the atoms is totally described by the interaction strength g . In any of these cases, the system can be described by the coupling hamiltonian of eqn.2. The transfer of squeezing clearly depends on the interaction strength or in other words on the ratio of it with the frequency difference between the oscillators. The squeezing generated in the second mode is shown to be different in nature than single mode squeezing. We have also shown that the photon statistics of the affected mode can be made sub- or super-Poissonian to generate antibunched or bunched photon beam. The procedure given here could help in generating and/or controlling nonclassical states of radiation for applications in quantum optics.

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